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TEMPERATURE OF A CYLINDRICAL CAVITY WALL HEATED BY A PERIODIC FLUX

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Abstract—Response of the surface temperature of an initially uniform temperature material space bounded internally at radius $r = a$ and heated with a flux $Q = \bar{Q}\sin(\omega t+\varepsilon)$ is studied. This is done with the use of Duhamel's integral together with a previously obtained solution to the constant step heating problem. Application of the Euler transformation to short time expansions and the use of various asymptotic expansions result in analytic solution estimates useful for different ranges of ω and t. Together these estimates can be used to predict the entire history of the surface temperature for arbitrary ω and ε. They can easily be used to compute the $r = a$ surface temperature history under conditions of arbitrary periodic heating.

NOMENCLATURE

A, amplitude of θ_s in quasi-steady state;
a, radius of cavity surface;
C, $\exp(\gamma)$;
Ci, cosine integral;
$C^{(2)}$, Fresnel cosine integral;
$E^{(p)}$, multiple Euler transformation;
$e_m^{(p)}$, terms of transformed series;
d_m, coefficients in $\bar{\theta}_s$ expansion;
F_0, small to moderate τ estimate of $\bar{\theta}_s$;
F_∞, large τ estimate of $\bar{\theta}_s$;
f, g, functions related to sine and cosine integrals;
I_c, I_s, integrals used in definition of θ_s response;
$I_c^{(m)}$, terms in a series for estimating I_c;
$I_s^{(m)}$, terms in a series for estimating I_s;
K, a constant;
k, thermal conductivity;
p, number of times Euler transformation is taken;
Q, a periodic heat flux;
$\bar{Q}$, a constant heat flux;
r, radius;
$S^{(2)}$, Fresnel sine integral;
Si, sine integral;
T, temperature;
T_0, initial temperature;
T_s, surface temperature;
t, time.

Greek symbols

α, thermal diffusivity;
Γ, gamma function;
γ, Euler's constant;
ε, phase of Q;
ζ, a value of $\Omega\tau$ used for matching;
θ, dimensionless temperature;
θ_s, dimensionless surface temperature;
$\bar{\theta}$, dimensionless temperature field resulting from constant $\bar{Q}$ heating;
$\bar{\theta}_s$, dimensionless temperature of cavity wall heated with constant $\bar{Q}$;
λ_m, coefficients in θ_s expansion;
ρ, r/a;
τ, dimensionless time;
ϕ, phase lag between Q and θ_s;
Φ, dimensionless surface heat flux;
ξ, dummy variable;
Ω, dimensionless ω;
ω, frequency.

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1. INTRODUCTION

THIS paper studies the temperature field of a region bounded internally by a cylinder of radius $r = a$, where the region is initially at uniform temperature, T_0, and where at time $t = 0$ the cavity wall $r = a$ is heated by an oscillating heat flux Q given by

$$Q = \bar{Q}\sin(\omega t+\varepsilon) = \bar{Q}\sin(\Omega\tau+\varepsilon). \tag{1}$$

Here the dimensionless frequency, Ω, and time, τ, are defined as:

$$\Omega = \frac{\omega a^2}{\alpha}; \quad \tau = \frac{\alpha t}{a^2} \tag{2}$$

where α is the thermal diffusivity of the material. Attention is confined to analytic estimates of the cavity wall temperature history.

The estimates obtained—there are four in all—together are found to uniformly cover the entire range $0 < \tau$ and $0 < \Omega$. We define the dimensionless temperature field θ and dimensionless surface temperature θ_s as

$$\left.\begin{aligned} \theta &= 2(T-T_0)k/(\bar{Q}a) = \theta(\rho,\tau;\Omega,\varepsilon) \\ \theta_s &= \theta(1,\tau;\Omega,\varepsilon) = \theta_s(\tau;\Omega,\varepsilon) \end{aligned}\right\} \tag{3}$$

where T is the temperature field, $\rho = r/a$, and k is the thermal conductivity of the material. Then the four separate results referred to are:

Transient response for arbitrary Ω

$$\theta_s, \quad \tau \text{ fixed} \geqslant 0, \quad \Omega \text{ arbitrary}. \tag{4a}$$

Asymptotic approach to quasi-steady state in the limit of large Ω

$$\lim_{\Omega\to\infty} \theta_s \quad 0 < \tau \text{ fixed}. \tag{4b}$$

Asymptotic approach to quasi-steady state for fixed, nonzero Ω

$$\lim_{\Omega\tau\to\infty} \theta_s \quad 0 < \Omega \text{ fixed}. \tag{4c}$$

Asymptotic approach to quasi-steady state in the limit of small Ω

$$\lim_{\Omega\to 0} \theta_s \quad 0 < \Omega\tau \text{ fixed}. \tag{4d}$$

The utility of these results cover every possible application of the specified periodic surface heat flux problem (where the solid exterior to the cavity can be accurately modeled with uniform and constant properties k and α). Thus, for example, the large Ω estimates of (4a) and (4b) would find application in predicting internal wall temperatures in rapid fire, thick walled gun barrels (i.e. where the temperature field resulting from the fluctuating component of the periodic heating was negligible at the exterior wall of the barrel) where some useful modeling of the internal firing phenomenology yielded a periodic wall flux boundary condition [1].* On the other extreme, the small Ω estimates of (4a) and (4d) would find application in cylindrical cavity earth heat exchanger problems where the fundamental period of the periodic heating was based on the annual cycle. The moderate Ω estimates of (4a) and (4c) complete the general solution to the stated problem. These would find application, for example, in the latter earth heat exchanger problem where the details of the daily periodic heating cycle were of interest.

2. STATEMENT OF THE PROBLEM

The temperature field, θ, on and exterior to the cavity wall is governed by the following boundary value problem:

$$\begin{aligned}
&\frac{\partial\theta}{\partial\tau} = \frac{\partial^2\theta}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\theta}{\partial\rho} \\
&\rho > 1, \quad \tau = 0: \theta = 0 \\
&\rho = 1, \quad \tau > 0: -\frac{\partial\theta}{\partial\rho} = 2\Phi(\tau) = 2\sin(\Omega\tau+\varepsilon) \\
&\rho \to \infty, \quad \tau > 0: \theta \to 0.
\end{aligned} \tag{5}$$

Our goal here is an analytic estimate of $\theta_s(\tau; \Omega, \varepsilon)$ for arbitrary values of τ, Ω, and ε.

Using Duhamel's integral we can represent our solution as

$$\theta_s = I_c \sin(\Omega\tau+\varepsilon) - I_s\cos(\Omega\tau+\varepsilon) \tag{6}$$

where

$$I_c = \int_0^\tau \cos\Omega\xi \frac{\partial\bar{\theta}_s}{\partial\xi}\,d\xi; \quad I_s = \int_0^\tau \sin\Omega\xi \frac{\partial\bar{\theta}_s}{\partial\xi}\,d\xi.$$

Here, $\bar{\theta}_s = \bar{\theta}(\rho = 1, \tau; \Omega, \varepsilon)$, where $\bar{\theta}$ is the solution to the problem of equations (5) with $\Phi(\tau) = 1$. A solution for $\bar{\theta}$ has been obtained in [2] and the results for $\bar{\theta}_s$ have been extended in [3]. Using the results of the latter reference we find that $\bar{\theta}_s$ can be estimated from:

$$\begin{aligned}
\bar{\theta}_s = F_0(\tau) &= E^{(p)}\left[\sum_{m=1}^{M} d_m \tau^{m/2} \Big/ \Gamma(1+m/2)\right] \\
&= E^{(p)}\left[\sum_{m=1}^{M} \lambda_m \tau^{m/2}\right] = \sum_{m=1}^{M} e_m^{(p)}(\tau), \\
&\qquad 0 \leqslant \tau \leqslant O(10),
\end{aligned} \tag{7}$$

where

$$\begin{aligned}
d_1 &= 2, \\
d_{m+1} &= \frac{(-1)^m[(2m-1)!]^2}{2^{5m-2}m[(m-1)!]^3} \\
&\quad + \sum_{n=1}^{m} \frac{(-1)^n(2n+1)!(2n-2)!}{2^{5n-1}[(n-1)!]^3 n^2} d_{m-n+1}, \; m > 0,
\end{aligned}$$

and, for large τ, from:

$$\begin{aligned}
\bar{\theta}_s = F_\infty(\tau) &= \ln(4\tau/C) + \ln(4\tau/C)/(2\tau) + 1/(2\tau) \\
&\quad - 3\ln^2(4\tau/C)/(16\tau^2) - \ln(4\tau/C)/(16\tau^2) \\
&\quad + (\pi^2+3)/(32\tau^2) + O(\ln^3\tau/\tau^3).
\end{aligned} \tag{8}$$

In equation (7), $\Gamma(x)$ is the gamma function and $\ln C = \gamma = 0.57722\ldots$ is Euler's constant. Also, $E^{(p)}$ represents the pth Euler transformation of the M term partial sum $\Sigma\,\lambda_m\tau^{m/2}$. Thus, according to the definition of the Euler transformation in [3, 4], and again referring to equation (7),

$$\begin{aligned}
e_m^{(0)} &= \lambda_m \tau^{m/2}, \\
e_m^{(p)} &= \frac{1}{2^m}\sum_{n=1}^{m} \frac{(m-1)!}{(m-n)!(n-1)!} e_{m-n+1}^{(p-1)}, \quad p > 0.
\end{aligned} \tag{9}$$

The series $\Sigma\,\lambda_m\tau^{m/2}$ of equation (7) is the small τ expansion of $\bar{\theta}_s$. It has been shown in [3] that, whereas this small term expansion is only useful in a limited range $0 \leqslant \tau \ll 1$, the application of the Euler transformation to this expansion as per equation (7) (with $p > 0$) allows one to extract accurate estimates for $\bar{\theta}_s$ well into the moderate τ region. [For example, by using equation (7) with $p = 3$ and $M = 19$, $\bar{\theta}_s(\tau = 30)$ can be estimated to within 1% of the exact result.] Accordingly, equations (7) and (8) together can be used to estimate $\bar{\theta}_s$ in the entire range $0 < \tau$, and to do this within a maximum error of 1%

In all applications of the Euler transformation here and below, it appears that optimum values of M and p (i.e. those values which result in the most accurate estimate) will satisfy $M \leqslant 25$ and $p \leqslant 4$. The optimum values of M and p for each individual computation would be those which yielded latter terms (say the last two terms) in the truncated $\Sigma\,e_m^{(p)}$ series which were smallest in absolute value compared to the summed series. Further remarks on the suggested computational procedure for the Euler transformed series which are applicable throughout this work are provided in [3].

*Numbers in brackets refer to the list of references at the end of this paper.

The plan here is to use equations (7) and (8) in equation (6) thereby obtaining analytic results for θ_s useful for all $0 < \tau$ and arbitrary Ω and ε. Each of the four results indicated above in (4) will be investigated in the following four sections. Finally, the last section will present a summary of results along with illustrative calculations of the θ_s response for large, moderate and small values of Ω.

3. RESULTS FOR θ_s, τ FIXED ≥ 0, Ω ARBITRARY

As indicated above, equation (7) can be used to accurately estimate $\bar{\theta}_s$ in the range $\tau = O(10)$ [3]. Accordingly, we expect that this equation can be directly utilized in equation (6) to obtain an analytic result for θ_s which is also useful within this τ range. Such a result would yield estimates for θ_s at least during some initial interval of its transient response. Moreover, this would be true irregardless of the size of Ω.

Instead of direct substitution of equation (7) into equation (6) it is more convenient and equivalent to first replace $\bar{\theta}_s$ in equation (6) by its small τ expansion, $\Sigma\lambda_m\tau^{m/2}$, take the appropriate term by term derivatives of this, perform the indicated multiplication (by $\cos\Omega\xi$ or $\sin\Omega\xi$) and term by term integration, and to finally take the multiple Euler transformation of the result. Doing all this leads to the following results for I_c and I_s:

$$\begin{Bmatrix} I_c \\ I_s \end{Bmatrix} = E^{(p)}\left[\sum_{m=1}^{M} \frac{m\lambda_m}{2\Omega^{m/2}} \begin{Bmatrix} I_c^{(m)}(\Omega\tau) \\ I_s^{(m)}(\Omega\tau) \end{Bmatrix}\right], \tag{10}$$

where

$$I_c^{(1)} = (2\pi)^{1/2}C^{(2)}(\Omega\tau), \quad I_s^{(1)} = (2\pi)^{1/2}S^{(2)}(\Omega\tau),$$
$$I_c^{(2)} = \sin(\Omega\tau), \qquad I_s^{(2)} = 1-\cos(\Omega\tau),$$
$$\begin{Bmatrix} I_c^{(3)} \\ I_s^{(3)} \end{Bmatrix} = (\Omega\tau)^{1/2}\begin{Bmatrix} \sin(\Omega\tau) \\ -\cos(\Omega\tau) \end{Bmatrix} - \frac{(2\pi)^{1/2}}{2}\begin{Bmatrix} S^{(2)}(\Omega\tau) \\ -C^{(2)}(\Omega\tau) \end{Bmatrix},$$
$$\begin{Bmatrix} I_c^{(4)} \\ I_s^{(4)} \end{Bmatrix} = (\Omega\tau)\begin{Bmatrix} \sin(\Omega\tau) \\ -\cos(\Omega\tau) \end{Bmatrix} + \begin{Bmatrix} \cos(\Omega\tau)-1 \\ \sin(\Omega\tau) \end{Bmatrix},$$
$$\begin{Bmatrix} I_c^{(m)} \\ I_s^{(m)} \end{Bmatrix} = (\Omega\tau)^{(m/2-1)}\begin{Bmatrix} \sin(\Omega\tau) \\ -\cos(\Omega\tau) \end{Bmatrix} + \frac{(m-2)}{2}(\Omega\tau)^{(m/2-2)}\begin{Bmatrix} \cos(\Omega\tau) \\ \sin(\Omega\tau) \end{Bmatrix} - \frac{(m-2)(m-4)}{4}\begin{Bmatrix} I_c^{(m-4)} \\ I_s^{(m-4)} \end{Bmatrix}, \quad m > 4.$$

Here $C^{(2)}(x)$ and $S^{(2)}(x)$ are Fresnel integrals defined as [5]:

$$C^{(2)}(x) = \frac{1}{(2\pi)^{1/2}}\int_0^x \frac{\cos\xi}{\xi^{1/2}}\,d\xi, \qquad S^{(2)}(x) = \frac{1}{(2\pi)^{1/2}}\int_0^x \frac{\sin\xi}{\xi^{1/2}}\,d\xi. \tag{11}$$

As suggested above, equation (6) will be used together with equation (10) to estimate our solution for θ_s for arbitrary Ω and ε and for $\tau \leq O(10)$. In this range it has been shown in [3] that (with predictable accuracy) equation (7) can be used to estimate $\bar{\theta}_s$ to within an error of the order of 1%, and we expect the situation to be carried over to our present solution for θ_s.

Taking the limit $\Omega \to \infty$ of the above results for fixed $\Omega\tau \geq 0$ leads us to the special case of heating of a halfspace (i.e. $a \to \infty$, $\omega t = \Omega\tau$ fixed). Under this limit we obtain results for θ_s from the leading terms of equation (10) (with $p = 0$). Using these with equation (6) we obtain

$$\lim_{\substack{\Omega\to\infty \\ 0\leq\Omega\tau \text{ fixed}}} \frac{\Omega^{1/2}\theta_s}{2} = \lim_{\substack{\Omega\to\infty \\ 0\leq\Omega\tau \text{ fixed}}} \frac{k}{\bar{Q}}\left(\frac{\omega}{\alpha}\right)^{1/2}(T_s - T_0)$$
$$= 2^{1/2}[C^{(2)}(\omega t)\sin(\omega t+\varepsilon) - S^{(2)}(\omega t)\cos(\omega t+\varepsilon)] + O(1/\Omega^{1/2}) \tag{12}$$

where T_s is the dimensional surface temperature. By representing our results in the latter form we exhibit the requirement that the temperature field of the heated halfspace or "large" radius cylindrical cavity surface must be independent of the radial dimension, a. The result of equation (12) is consistent with the halfspace temperature field solution given in an integral form in [2].

4. RESULTS FOR θ_s, τ FIXED > 0, $\Omega \to \infty$

The result of the previous section is expected to be useful up to large natural diffusion time τ of the material. When Ω is large this means that equation (10) will be useful from $\tau = 0$ and on through many periods of the heating, i.e. up to values of $\Omega\tau$ which will be many multiples of 2π. In other words, for large Ω we expect that equation (10) will be useful well into the time of quasi-steady state temperature oscillation which must eventually be developed within the solid and on the cavity surface. Accordingly, with fixed τ we can take the limit as $\Omega \to \infty$ of I_c and I_s per equation (10) and hope to extract θ_s results from equation (6) which characterize its asymptotic approach to quasi-steady state. Doing so we eventually obtain the following results:

$$\lim_{\substack{\Omega\to\infty \\ 0<\tau \text{ fixed}}} I_c = \left(\frac{2}{\Omega}\right)^{1/2} + \frac{1}{\Omega}\frac{d\bar{\theta}_s}{d\tau}\sin(\Omega\tau) - \frac{3}{2(2\Omega)^{3/2}} + \frac{1}{\Omega^2}\left[\frac{3}{4} + \frac{d^2\bar{\theta}_s}{d\tau^2}\cos(\Omega\tau)\right] - \frac{63}{16(2\Omega)^{5/2}} - \frac{1}{\Omega^3}\frac{d^3\bar{\theta}_s}{d\tau^3}\sin(\Omega\tau) + \frac{105\pi^{1/2}\lambda_7}{2(2\Omega)^{7/2}} + O\left(\frac{1}{\Omega^4}\right) \tag{13}$$

$$\lim_{\substack{\Omega\to\infty \\ 0<\tau \text{ fixed}}} I_s = \left(\frac{2}{\Omega}\right)^{1/2} - \frac{1}{\Omega}\left[1 + \frac{d\bar{\theta}_s}{d\tau}\cos(\Omega\tau)\right] + \frac{3}{2(2\Omega)^{3/2}} + \frac{1}{\Omega^2}\frac{d^2\bar{\theta}_s}{d\tau^2}\sin(\Omega\tau) - \frac{63}{16(2\Omega)^{5/2}} + \frac{1}{\Omega^3}\left[-6\lambda_6 + \frac{d^3\bar{\theta}_s}{d\tau^3}\cos(\Omega\tau)\right] - \frac{105\pi^{1/2}\lambda_7}{2(2\Omega)^{7/2}} + O\left(\frac{1}{\Omega^4}\right). \tag{14}$$

Using these results in equation (6) we obtain

$$\lim_{\substack{\Omega\to\infty \\ 0<\tau \text{ fixed}}} \theta_s = \frac{2}{\Omega^{1/2}}\sin(\Omega\tau+\varepsilon-\pi/4)+\frac{1}{\Omega}\left[\cos(\Omega\tau+\varepsilon)+\frac{d\bar{\theta}_s}{d\tau}\cos\varepsilon\right]-\frac{3}{4\Omega^{3/2}}\sin(\Omega\tau+\varepsilon+\pi/4)+\frac{1}{\Omega^2}\left[\tfrac{3}{4}\sin(\Omega\tau+\varepsilon)+\frac{d^2\bar{\theta}_s}{d\tau^2}\sin\varepsilon\right]-\frac{63}{64\Omega^{5/2}}\sin(\Omega\tau+\varepsilon-\pi/4)+\frac{1}{\Omega^3}\left[6\lambda_6\cos(\Omega\tau+\varepsilon)-\frac{d^3\bar{\theta}_s}{d\tau^3}\cos\varepsilon\right]+\frac{105\pi^{1/2}\lambda_7}{16\Omega^{7/2}}\sin(\Omega\tau+\varepsilon+\pi/4)+O\left(\frac{1}{\Omega^4}\right). \tag{15}$$

Depending on the value of τ the value of $d\bar{\theta}_s/d\tau$, $d^2\bar{\theta}_s/d\tau^2$ and $d^3\bar{\theta}_s/d\tau^3$ in the above can be computed from either equation (7) or equation (8), i.e. from:

$$\frac{d\bar{\theta}_s}{d\tau}=\frac{dF_0}{d\tau}=\frac{1}{2\tau}E^{(p)}\left[\sum_{m=1}^{M} m\lambda_m\tau^{m/2}\right],\quad \frac{d^2\bar{\theta}_s}{d\tau^2}=\frac{d^2F_0}{d\tau^2},\quad \frac{d^3\bar{\theta}_s}{d\tau^3}=\frac{d^3F_0}{d\tau^3}, \tag{16}$$

or from

$$\frac{d\bar{\theta}_s}{d\tau}=\frac{dF_\infty}{d\tau}=\frac{1}{\tau}\left[1-\frac{1}{2\tau}\ln(4\tau/C)+\frac{3}{8\tau^2}\ln^2(4\tau/C)-\frac{1}{4\tau^2}\ln(4\tau/C)-\frac{(\pi^2+4)}{16\tau^2}+O\left(\frac{\ln^3\tau}{\tau^3}\right)\right],\quad \frac{d^2\bar{\theta}_s}{d\tau^2}=\frac{d^2F_\infty}{d\tau^2},\quad \frac{d^3\bar{\theta}_s}{d\tau^3}=\frac{d^3F_\infty}{d\tau^3}. \tag{17}$$

Three features of the results of equations (13)–(15) are noteworthy. First, they appear to be useful (when $\Omega \gg 1$) for all τ where $d\bar{\theta}_s/d\tau$, $d^2\bar{\theta}_s/d\tau^2$ and $d^3\bar{\theta}_s/d\tau^3$ are $O(1)$, e.g. at least for $\tau > 1$. The region of applicability of these results therefore overlap with the region of applicability of the results of the previous section. Second, when $\tau \gg 1$ and therefore when equations (17) are useful, our analytic result of equation (15) is particularly simple to use. Finally, in the quasi-steady state ($\tau\to\infty$ and therefore $d\bar{\theta}_s/d\tau$, $d^2\bar{\theta}_s/d\tau^2$ and $d^3\bar{\theta}_s/d\tau^3 \to 0$) the lag in phase, ϕ, between the oscillating surface temperature and the oscillating heat flux, along with the amplitude, A, of θ_s, can be computed from:

$$\lim_{\tau\to\infty}\theta_s = A\sin(\Omega\tau+\varepsilon-\phi) \tag{18}$$

where

$$A=\lim_{\tau\to\infty}[I_s^2+I_c^2]^{1/2},\quad \phi=\lim_{\tau\to\infty}\tan^{-1}(I_s/I_c)$$

and where under the particular limit of this section

$$\lim_{\Omega\to\infty} A=\frac{2}{\Omega^{1/2}}\left[1-\frac{1}{4}\left(\frac{2}{\Omega}\right)^{1/2}+\frac{1}{32}\left(\frac{2}{\Omega}\right)+\frac{7}{128}\left(\frac{2}{\Omega}\right)^{3/2}-\frac{189}{2048}\left(\frac{2}{\Omega}\right)^{4/2}+O\left(\frac{2}{\Omega}\right)^{5/2}\right]$$
$$\lim_{\Omega\to\infty}\phi=\tan^{-1}\left[1-\frac{1}{2}\left(\frac{2}{\Omega}\right)^{1/2}+\frac{3}{8}\left(\frac{2}{\Omega}\right)-\frac{9}{32}\left(\frac{2}{\Omega}\right)^{3/2}+\frac{21}{128}\left(\frac{2}{\Omega}\right)^{4/2}+O\left(\frac{2}{\Omega}\right)^{5/2}\right]. \tag{19}$$

The value of $\lim_{\Omega\to\infty}\phi=\pi/4$ is in required agreement with the results of periodic heating of a halfspace [2].

5. RESULTS FOR θ_s, $\Omega\to 0$, $0<\Omega\tau$ FIXED

In the range of moderate to small Ω the results of Section 3, useful up to moderate values of τ, cannot be expected to reveal the asymptotic approach to quasi-steady behavior of θ_s. This is, of course, especially true for small Ω problems since for these the Section 3 results will not even extend up to a complete period ($\Omega\tau = 2\pi$) of the heating.

In this section we will study the problem of the approach to quasi-steady state under the limit $\Omega\to 0$. The basic difference in the analysis here from that of Section 3 is that $d\bar{\theta}_s/d\tau$ in the integrals I_c and I_s of equation (6) must be estimated by a combination of the small to moderate τ results of equation (7) together with the large τ results of equation (8). Thus, here we need to compute

$$\begin{Bmatrix} I_c \\ I_s \end{Bmatrix}=\int_0^K\begin{Bmatrix}\cos\Omega\xi\\ \sin\Omega\xi\end{Bmatrix}\frac{dF_0}{d\xi}d\xi+\int_K^\tau\begin{Bmatrix}\cos\Omega\xi\\ \sin\Omega\xi\end{Bmatrix}\frac{dF_\infty}{d\xi}d\xi \tag{20}$$

where K is any moderate value such that both the F_0 and F_∞ representations of $\bar{\theta}_s(K)$ will yield reasonable estimates.

Using the results for $dF_0/d\tau$ and $dF_\infty/d\tau$ as per equations (16) and (17) in equation (20) the following results were eventually obtained:

$$\lim_{\substack{\Omega\to 0\\ 0<\Omega\tau\text{ fixed}}} I_c = -\ln\Omega+[Ci(\Omega\tau)+\ln(4/C^2)]-\tfrac{1}{2}\Omega\ln\Omega\left[\frac{\cos(\Omega\tau)}{\Omega\tau}+Si(\Omega\tau)\right]+O(\Omega) \tag{21}$$

$$\lim_{\substack{\Omega\to 0\\ 0<\Omega\tau\text{ fixed}}} I_s = Si(\Omega\tau)-\tfrac{1}{4}\Omega\ln^2\Omega-\tfrac{1}{2}\Omega\ln\Omega\times\left[-Ci(\Omega\tau)+\frac{\sin(\Omega\tau)}{\Omega\tau}-1-\ln(4/C^2)\right]+O(\Omega) \tag{22}$$

where the cosine and sine integrals, $Ci(x)$ and $Si(x)$, respectively, are defined as, [5]:

$$Ci(x)=-\int_x^\infty\frac{\cos\xi}{\xi}d\xi=\ln(xC)+\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{2n(2n)!}$$
$$Si(x)=\int_0^x\frac{\sin\xi}{\xi}d\xi=\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}. \tag{23}$$

Using the results of equations (21) and (22) in equation (6), we finally attain our present objective as follows:

$$\begin{aligned}\lim_{\substack{\Omega\to 0\\ 0<\Omega\tau \text{ fixed}}} \theta_s = &-\ln\Omega\sin(\Omega\tau+\varepsilon)\\ &+\left[\ln(4/C^2)\sin(\Omega\tau+\varepsilon)-\frac{\pi}{2}\cos(\Omega\tau+\varepsilon)-g(\Omega\tau)\sin\varepsilon+f(\Omega\tau)\cos\varepsilon\right]+\frac{1}{4}\Omega\ln^2\Omega\cos(\Omega\tau+\varepsilon)\\ &-\frac{1}{2}\Omega\ln\Omega\left[\frac{\sin\varepsilon}{\Omega\tau}+\{1+\ln(4/C^2)\}\cos(\Omega\tau+\varepsilon)+\frac{\pi}{2}\sin(\Omega\tau+\varepsilon)-g(\Omega\tau)\cos\varepsilon-f(\Omega\tau)\sin\varepsilon\right]+O(\Omega)\end{aligned} \quad (24)$$

where, [5]:

$$\begin{aligned} f(x) &= Ci(x)\sin x-\left[Si(x)-\frac{\pi}{2}\right]\cos x=\frac{1}{x}\left[1+O\left(\frac{1}{x^2}\right)\right]\\ g(x) &= -Ci(x)\cos x-\left[Si(x)-\frac{\pi}{2}\right]\sin x=\frac{1}{x^2}\left[1+O\left(\frac{1}{x^2}\right)\right].\end{aligned} \quad (25)$$

Further, with regard to the quasi-steady behaviour of θ_s as per equations (18), we find from equations (21) and (22) that

$$\lim_{\Omega\to 0} A = -\ln\Omega\left[1+2\ln(4/C^2)/\ln\Omega+\left\{\frac{\pi^2}{4}+\ln^2(4/C^2)\right\}\Big/\ln^2\Omega+\pi\Omega/4\right]^{1/2}[1+O(\Omega/\ln\Omega)] \quad (26)$$

$$\lim_{\Omega\to 0}\phi = \tan^{-1}\left[-\frac{\pi}{2\ln\Omega}\left\{\frac{1-\dfrac{\Omega\ln^2\Omega}{2\pi}+\dfrac{1}{\pi}[1-\ln(4/C^2)]\Omega\ln\Omega}{1+\ln(4/C^2)/\ln\Omega+\pi\Omega/4}\right\}\left[1+O\left(\frac{1}{\Omega}\right)\right]\right]. \quad (27)$$

As is evident from these last results, and from equation (25), the quasi-steady amplitude of θ_s becomes unbounded and the phase lag, ϕ, goes to zero as $\Omega\to 0$.

6. RESULTS FOR θ_s, $\Omega\tau\to\infty$, $0<\Omega$ FIXED

For all $0<\Omega$ the results of Section 3 provide us with an estimate of the θ_s history up to moderate values of τ. For large Ω the results of Section 4 provide us with a more readily usable estimate of the θ_s history in the range where θ_s approaches and achieves quasi-steady state (i.e. in the large $\Omega\tau$ range). For small Ω the approach to quasi-steady state of θ_s will occur within the moderate $\Omega\tau$ range (e.g. within the first few cycles of heating). Estimates of the θ_s history in that range are provided by the results of Section 5. In the range of moderate Ω we still seek a solution describing the history of θ_s during its approach to quasi-steady state. We will obtain this here from a representation for θ_s in the limit of $\Omega\tau\to\infty$ with $0<\Omega$ fixed. To this end we compute I_c and I_s according to

$$\begin{Bmatrix} I_c(\Omega\tau;\Omega)\\ I_s(\Omega\tau;\Omega)\end{Bmatrix}=\begin{Bmatrix} I_c(\infty;\Omega)\\ I_s(\infty;\Omega)\end{Bmatrix}-\int_\tau^\infty\begin{Bmatrix}\cos\Omega\xi\\ \sin\Omega\xi\end{Bmatrix}\frac{d\bar{\theta}_s}{d\xi}d\xi. \quad (28)$$

For large τ the latter integrals have been estimated with the use of the F_∞ representation of $\bar{\theta}_s$ as per equations (8) and (17). Using the result in equation (28) we find

$$\begin{aligned}\lim_{\Omega\tau\to\infty}\begin{Bmatrix} I_c(\Omega\tau;\Omega)\\ I_s(\Omega\tau;\Omega)\end{Bmatrix}=&\begin{Bmatrix} I_c(\infty;\Omega)\\ I_s(\infty;\Omega)\end{Bmatrix}+\frac{1}{\Omega\tau}\begin{Bmatrix}\sin(\Omega\tau)\\ -\cos(\Omega\tau)\end{Bmatrix}-\frac{\Omega}{2}\frac{\ln(\Omega\tau)}{(\Omega\tau)^2}\begin{Bmatrix}\sin(\Omega\tau)\\ -\cos(\Omega\tau)\end{Bmatrix}\\ &-\frac{1}{(\Omega\tau)^2}\left[\begin{Bmatrix}\cos(\Omega\tau)\\ \sin(\Omega\tau)\end{Bmatrix}+\frac{\Omega}{2}\ln\left(\frac{4}{C\Omega}\right)\begin{Bmatrix}\sin(\Omega\tau)\\ -\cos(\Omega\tau)\end{Bmatrix}\right]+\frac{3\Omega^2}{8}\frac{\ln^2(\Omega\tau)}{(\Omega\tau)^3}\begin{Bmatrix}\sin(\Omega\tau)\\ -\cos(\Omega\tau)\end{Bmatrix}\\ &+\frac{\ln(\Omega\tau)}{(\Omega\tau)^3}\left[\Omega\begin{Bmatrix}\cos(\Omega\tau)\\ \sin(\Omega\tau)\end{Bmatrix}+\left\{\frac{3\Omega^2}{4}\ln\left(\frac{4}{C\Omega}\right)-\frac{\Omega^2}{4}\right\}\begin{Bmatrix}\sin(\Omega\tau)\\ -\cos(\Omega\tau)\end{Bmatrix}\right]\\ &+\frac{1}{(\Omega\tau)^3}\left[\left\{\Omega\ln\left(\frac{4}{C\Omega}\right)-\frac{\Omega}{2}\right\}\begin{Bmatrix}\cos(\Omega\tau)\\ \sin(\Omega\tau)\end{Bmatrix}\right.\\ &\qquad\left.+\left\{\frac{3\Omega^2}{8}\ln^2\left(\frac{4}{C\Omega}\right)-\frac{\Omega^2}{4}\ln\left(\frac{4}{C\Omega}\right)-\frac{\Omega^2}{16}(\pi^2+4)-2\right\}\begin{Bmatrix}\sin(\Omega\tau)\\ -\cos(\Omega\tau)\end{Bmatrix}\right]\\ &+O\left[\frac{\ln^3(\Omega\tau)}{(\Omega\tau)^4}\right].\end{aligned} \quad (29)$$

Further, using these results in equation (6) we finally obtain

$$\begin{aligned}\lim_{\substack{\Omega\tau\to\infty\\ 0<\Omega\text{ fixed}}} \theta_s = {} & A\sin(\Omega\tau+\varepsilon-\phi)+\left[\frac{1}{\Omega\tau}-\frac{\Omega}{2}\frac{\ln(\Omega\tau)}{(\Omega\tau)^2}\right]\cos\varepsilon \\ & -\frac{1}{(\Omega\tau)^2}\left[\sin\varepsilon+\frac{\Omega}{2}\ln\left(\frac{4}{C\Omega}\right)\cos\varepsilon\right]+\frac{3\Omega^2}{9}\frac{\ln^2(\Omega\tau)}{(\Omega\tau)^3}\cos\varepsilon+\frac{\ln(\Omega\tau)}{(\Omega\tau)^3}\left[\Omega\sin\varepsilon+\frac{\Omega^2}{4}\left\{3\ln\left(\frac{4}{C\Omega}\right)-1\right\}\cos\varepsilon\right] \\ & +\frac{1}{(\Omega\tau)^3}\left[\Omega\left\{\ln\left(\frac{4}{C\Omega}\right)-\frac{1}{2}\right\}\sin\varepsilon+\left\{\frac{3\Omega^2}{8}\ln^2\left(\frac{4}{C\Omega}\right)-\frac{\Omega^2}{4}\ln\left(\frac{4}{C\Omega}\right)-\frac{\Omega^2}{16}(\pi^2+4)-2\right\}\cos\varepsilon\right] \\ & +O\left[\frac{\ln^3(\Omega\tau)}{(\Omega\tau)^4}\right]. \end{aligned} \tag{30}$$

Here again A and ϕ can be obtained from $I_c(\infty;\Omega)$ and $I_s(\infty;\Omega)$ according to equation (18). The only problem remaining is that these latter two functions of Ω are not yet available to us. This situation can be rectified, and our present estimate can therefore be completed by matching, at some $\Omega\tau = \zeta$, the above results of equation (29) with the results of I_c and I_s obtained in equations (10). Thus we obtain

$$\begin{Bmatrix} I_c(\infty;\Omega) \\ I_s(\infty;\Omega)\end{Bmatrix} = E^{(p)}\left[\sum_{m=1}^{M}\frac{m\lambda_m}{2\Omega^{m/2}}\begin{Bmatrix} I_c^{(m)}(\zeta) \\ I_s^{(m)}(\zeta)\end{Bmatrix}\right]-\frac{1}{\zeta}\begin{Bmatrix}\sin\zeta \\ -\cos\zeta\end{Bmatrix}+\frac{\Omega}{2}\frac{\ln\zeta}{\zeta^2}\begin{Bmatrix}\sin\zeta \\ -\cos\zeta\end{Bmatrix}+\ldots \tag{31}$$

where terms up to $O(\ln^3\zeta/\zeta^4)$ are available but have not been displayed. The above value of ζ must be compatible with the useful ranges of the expansions that were used and it should be chosen so as to minimize the anticipated errors. Suggested values for ζ in the above are $\zeta = 2\pi$ for the computation of $I_c(\infty;\Omega)$ and $\zeta = 3\pi/2$ for that of $I_s(\infty;\Omega)$. It is further recommended that $I_c(\infty;\Omega)$ be computed from equation (29) for $0.14 \leqslant \Omega < 4$ and that $I_s(\infty;\Omega)$ be computed from equation (29) for $0.05 \leqslant \Omega < 4$. For values of Ω above and below these ranges of Ω, the results of Sections 4 and 5, respectively, should be used.

Following the above guidelines $I_c(\infty;\Omega)$ and $I_s(\infty;\Omega)$ have been computed in the range $10^{-2} < \Omega < 10^2$ (with a maximum error estimated to be of the order of 1%) and plotted in Fig. 1. Using equation (18) these results have been used to obtain the corresponding values for A and ϕ. These are also plotted in Fig. 1.

7. SUMMARY OF RESULTS AND SOME ILLUSTRATIVE COMPUTATIONS

The results of the last four sections were used to develop analytic estimates for θ_s in different ranges of the τ, Ω field. Of these estimates the suggested best choice as a function of the value of Ω and $\Omega\tau$ are summarized in Fig. 2. As noted in that figure, there is one region that is not covered by our four separate results, namely $30\Omega < \Omega\tau < 0.5$. Within this region and for a given Ω, θ_s can be estimated from equation (24) where terms only up to and including the $O(\Omega\ln^2\Omega)$ term are used. (Indeed, a careful investigation of such

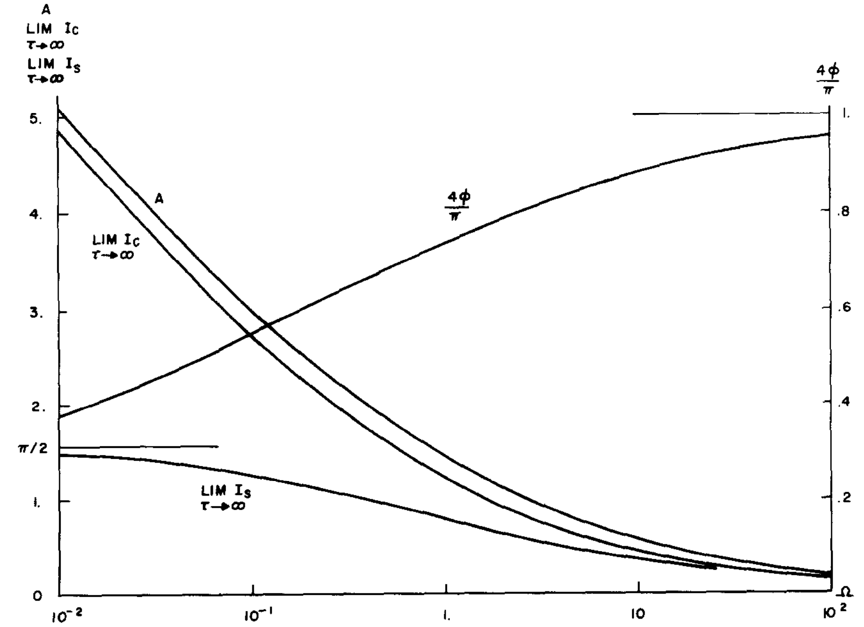

FIG. 1. Plot of $\lim_{\tau\to\infty} I_c$, $\lim_{\tau\to\infty} I_s$, A and ϕ as functions of Ω.

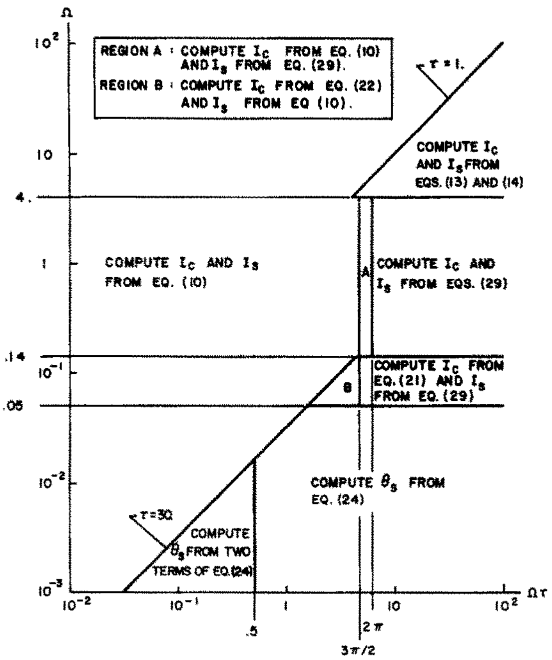

FIG. 2. Recommended computational procedure for θ_s or for I_c and I_s.

an estimate in the limit of small $\Omega\tau$ corroborates large τ results obtained in [3] for the case of cavities heated with fluxes that are linear in τ.) Following the recommendations of Fig. 2 it is estimated that the maximum error to be anticipated in the computed value of θ_s would be of the order of 1% of its quasi-steady amplitude A.

The quasi-steady behavior of θ_s as defined in equation (18) is determined by the values of $A(\Omega)$ and $\phi(\Omega)$. From the results of Sections 4–6 these functions are plotted in Fig. 1. The plotted values for A and ϕ can be reproduced from the appropriate estimates presented earlier.

To illustrate the general response of θ_s, appropriate computations for large (100), moderate (1), and small (1/100) values of Ω and for $\varepsilon = n\pi/4$, $n = 0, \pm 1, \ldots$, have been performed. The results of these computations are shown in Fig. 3 where the response in the range $0 \leqslant \Omega\tau \leqslant 6\pi$ are presented. The plots for $\Omega = 100$ are an accurate representation for the $\Omega \to \infty$ plots that would be obtained from the result of equation (12).

The various analytic estimates obtained herein can be used to readily compute transient and quasi-steady cylindrical cavity surface temperature histories under conditions of arbitrary periodic surface heating.

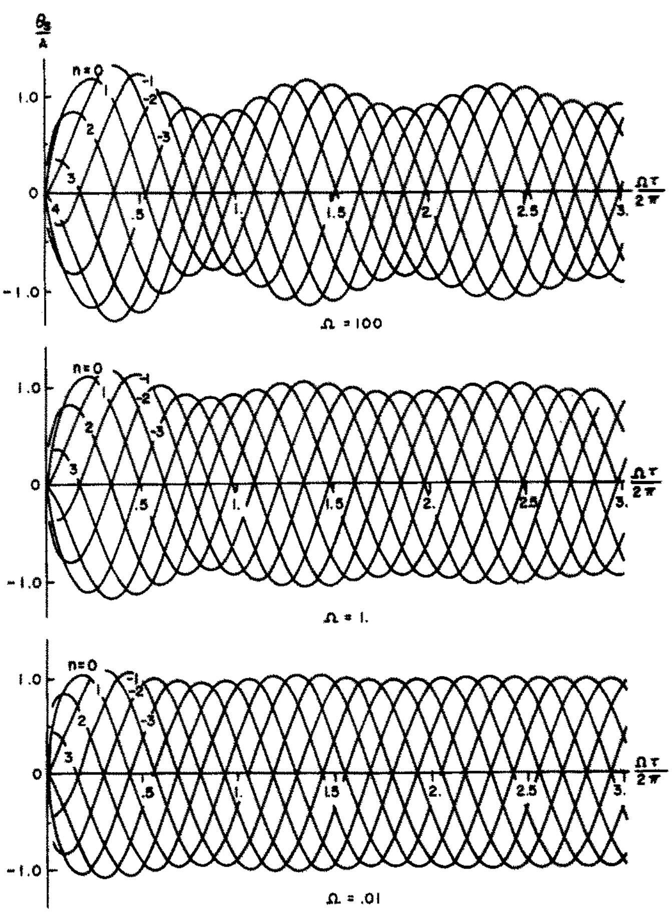

FIG. 3. Transient response of cavity surface temperature, θ_s, due to heat flux $Q \cdot \sin(\Omega\tau + n\pi/4)$ for $\Omega = 0.01$, 1.0 and 100.

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TEMPERATURE D'UNE CAVITE CYLINDRIQUE EN PAROI CHAUFFEE PAR UN FLUX PERIODIQUE

Résumé—On étudie la réponse de la température de surface d'un espace matériel limité intérieurement au rayon $r = a$ et chauffé par un flux $Q = \bar{Q}\sin(\omega t+\varepsilon)$. Le calcul est effectué à l'aide de l'intégrale de Duhamel connaissant la solution déjà obtenue du problème du créneau de chaleur. L'application de la transformation d'Euler aux développements valables pour les temps brefs et l'utilisation de divers résultats sur les développements asymptotiques dans les calculs de solutions analytiques ont été utiles dans différents domaines de variation de ω et t. De plus, ces estimations peuvent être employées pour déterminer l'histoire complète de la température de surface pour ω et ε arbitraires. Elles peuvent être aisément utilisées pour calculer l'histoire de la température à la surface $r = a$ sous des conditions arbitraires de chauffage périodique.

TEMPERATUR EINER, MIT EINEM PERIODISCHEN WÄRMESTROM AUFGEHEIZTEN WAND MIT ZYLINDRISCHEM HOHLRAUM

Zusammenfassung—Es wird der zeitliche Verlauf der Oberflächentemperatur eines Körpers mit anfänglich einheitlicher Temperatur und einem innenliegenden Hohlraum mit dem Radius $r = a$ untersucht, für den Fall, daß der Heizwärmestrom gemäß der Beziehung $Q = \bar{Q}\sin(\omega t+\varepsilon)$ zugeführt wird. Dabei wird das Duhamel-Integral zusammen mit einer zuvor ermittelten Lösung des Problems bei schrittweiser Beheizung verwendet. Die Anwendung der Euler-Transformation auf den Fall kurzer Zeiten und der Gebrauch verschiedener asymptotischer Entwicklungen ermöglicht analytische Näherungslösungen für verschiedene Bereiche von ω und t. Zusammen können diese Näherungslösungen zur Bestimmung des vollständigen zeitlichen Verlaufes der Oberflächentemperatur für beliebige ω und ε verwendet werden. Der zeitliche Verlauf der Temperatur an der Oberfläche $r = a$ kann auf einfache Weise bei beliebigen periodischen Heizbedingungen ermittelt werden.

ТЕМПЕРАТУРА СТЕНКИ ЦИЛИНДРИЧЕСКОЙ ПОЛОСТИ ПРИ ПЕРИОДИЧЕСКОМ НАГРЕВЕ

Аннотация — Изучалось влияние подводимого потока тепла в виде $Q = Q\sin(\omega t+\varepsilon)$ на температуру поверхности полости с внутренним диаметром $r = a$ при первоначально постоянной температуре поверхности полости. С этой целью использовался интеграл Дюамеля вместе с полученным ранее решением задачи с постоянным ступенчатым нагревом. С помощью преобразования Эйлера для разложения в случае малых времен, а также различных асимптотических разложений получены аналитические оценки, пригодные для различных диапазонов изменения ω и t. Совокупность этих оценок может быть использована для расчета временной зависимости температуры поверхности при произвольных значениях ω и ε, а также при расчете температуры поверхности $r = a$ в условиях произвольного периодического нагрева.